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The fermionic limit of the δ-function Bose gas: a pseudopotential approach

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Abstract

We use first-order perturbation theory near the fermionic limit of the δ-function Bose gas in one dimension (i.e. a system of weakly interacting fermions) to study three situations of physical interest. The calculation is done using a pseudopotential which takes the form of a two-body δ''-function interaction. The three cases considered are the behaviour of the system with a hard wall, with a point where the strength of the pseudopotential changes discontinuously, and with a region of finite length where the pseudopotential strength is non-zero (this is sometimes used as a model for a quantum wire). In all cases, we obtain exact expressions for the density to first order in the pseudopotential strength. The asymptotic behaviour of the densities is in agreement with the results obtained from bosonization for a Tomonaga–Luttinger liquid, namely, an interaction dependent power-law decay of the density far from the hard wall, a reflection from the point of discontinuity and transmission resonances for the interacting region of finite length. Our results provide a non-trivial verification of the Tomonaga–Luttinger liquid description of the δ-function Bose gas near the fermionic limit.

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1. Introduction

The δ-function Bose gas in one dimension has been studied extensively ever since Lieb and Liniger solved it using the Bethe ansatz [1]. However, the wavefunctions which one obtains from the Bethe ansatz are usually quite difficult to work with. It is therefore useful to examine other ways of studying the δ-function Bose gas. The Bose gas is characterized by a parameter c; for $c = \infty$, the model describes a system of non-interacting fermions. Since the physical properties of the system (such as the energy levels and thermodynamic properties) are known to vary continuously with c [1], it is natural to suppose that the Bose gas will be equivalent

to a system of weakly interacting fermions for large values of c, i.e. small values of $1/c$. With this in mind, we recently developed a way of doing perturbation theory near the limit $c = \infty$ [2]. This turned out to involve a two-body pseudopotential which takes the form of a δ''-function interaction acting between pairs of fermions. Let us denote the strength of the pseudopotential by a parameter g, which will be defined below in terms of the parameter $1/c$. In [2], we showed that this perturbative approach correctly reproduces the ground state energy up to order g^2. In this paper, we will apply the pseudopotential approach to three situations which cannot be solved using the Bethe ansatz; this will illustrate the power of this approach.

On general grounds, the low-energy and long-wavelength properties of a gapless system of interacting bosons or fermions in one dimension are believed to be described by a Tomonaga–Luttinger liquid (TLL) [3, 4]. A TLL is characterized by three quantities, the Fermi momentum k_F, the velocity of the low-energy excitations v (the dispersion relation of these excitations is given by $\omega = v|k|$) and the interaction parameter K. Once these parameters are known, the low-energy and long-wavelength properties of a TLL can be found by the technique of bosonization [3, 4]. In particular, K determines the exponents governing the long distance behaviour of various correlation functions. Clearly, it would be interesting to verify if the long distance properties of a δ-function Bose gas are indeed described by a TLL. We will show that the results obtained using the pseudopotential approach are in complete agreement with those expected from bosonization. Our study will therefore provide a non-trivial check of the pseudopotential approach, and will also confirm the expression for the parameter K in terms of g.

The plan of this paper is as follows. In section 2, we will introduce the δ-function Bose gas and the pseudopotential approach for doing perturbation theory near the fermionic limit $c = \infty$. In section 3, we will consider the δ-function Bose gas with a hard wall. We will obtain an expression for the density which is exact to first order in g. We will then find the asymptotic behaviour of the density far from the hard wall. The expression of the density will involve logarithmic factors, which, in section 6, will be recognized as being due to an interaction dependent power-law decay of the density which is characteristic of a TLL.

In section 4, we will consider a model in which the pseudopotential parameter g changes discontinuously at one point. We will again compute the density exactly to first order in g and show that there are oscillations which can be interpreted as being due to a reflection of the particles from that point. The amplitudes of reflection from the two sides are found to be equal to $\pm g$. The motivation for this section is the following. In [5], it was argued, on the basis of a bosonization analysis of a TLL, that if there is a discontinuity at one point in the strength of the two-body interaction between fermions, then it will necessarily cause some scattering from that point, even if there is no explicit one-body scattering potential present at that point. This statement has important implications for the conductance of a quantum wire (which is often modelled as a TLL with discontinuities in the interaction strengths, as discussed below). Now, bosonization is not a microscopic description of a system; it is only an effective description valid at long wavelengths. It is therefore instructive to verify in a specific microscopic model whether the above statement about scattering from a discontinuity is correct. The discussion in section 4 settles this issue by deriving the scattering amplitude in the δ-function Bose gas, and showing that it agrees with what is expected from the bosonization approach.

In section 5, we will extend the model of section 4 to the case of a region of finite length over which the pseudopotential has a non-zero strength. In section 6, we will discuss the TLL approach to the δ-function Bose gas, and will compute the Luttinger parameters K and v to first order in g. We will then use bosonization to compute the asymptotic behaviour of the density far from a hard wall [6] and from a point of discontinuity in the Luttinger parameters.

We will then see that these results agree precisely with those obtained in the previous sections. We will conclude in section 7 by pointing out some possible directions for future research.

2. The δ-function Bose gas

The δ-function Bose gas is defined by the Hamiltonian

$$H = -\frac{1}{2m}\sum_{1\leqslant i\leqslant N}\frac{\partial^2}{\partial x_i^2} + \frac{2c}{m}\sum_{1\leqslant i<j\leqslant N}\delta(x_i - x_j) \tag{1}$$

governing N identical bosons moving in one dimension. (We have set $\hbar = 1$). The interaction parameter c has the dimensions of (length)$^{-1}$. We will assume that c is non-negative, otherwise the thermodynamic limit ($N \to \infty$) is not well defined due to the presence of bound states with arbitrarily large negative energies.

We will consider two different boundary conditions for the calculations presented in this paper. In one case, the particles will be considered to be on a circle of circumference L with the wavefunctions satisfying periodic boundary conditions. In the other case, the particles will be considered to be in a box of length L with the wavefunctions vanishing if any of the particles coordinates is equal to 0 or L (this is called the hard wall condition). Since the particles are identical bosons, the wavefunctions must be completely symmetric. There are $N!$ possible orderings of the particle coordinates, given by $0 \leqslant x_{P_1} \leqslant x_{P_2} \leqslant \cdots \leqslant x_{P_N} \leqslant L$, where $(P_1, P_2, \ldots, P_N)$ is some permutation of the numbers $(1, 2, \ldots, N)$. If the wavefunctions are known for any one ordering, say $0 \leqslant x_1 \leqslant x_2 \leqslant \ldots x_N \leqslant L$, they are known for all other orderings by symmetry.

It is clear that the model describes non-interacting bosons for $c = 0$. For $c = \infty$, the wavefunctions vanish whenever any two particle coordinates coincide. We can then carry out the unitary transformation $\psi_P \to (-1)^P\psi_P$, where ψ_P denotes the wavefunction for the ordering P, and $(-1)^P$ denotes the sign of the permutation P. Under this transformation, the wavefunction becomes completely antisymmetric, i.e. fermionic. Thus $c = \infty$ denotes a system of non-interacting fermions. (Note that this unitary transformation of the wavefunctions is only allowed if $c = \infty$. At any other value of c, the symmetric wavefunctions do not vanish for $x_i = x_j$, and the transformation would produce antisymmetric wavefunctions which are discontinuous at those coincident points). Henceforth, we will refer to the model with $c = \infty$ as being non-interacting, and the model with c large as being weakly interacting in the fermionic sense (although it is strongly interacting in the bosonic sense).

In [2], we developed a way of doing perturbation theory near the fermionic limit in powers of $1/c$. We showed that the perturbation around $c = \infty$ is described by a two-particle interaction of the form

$$V = -\frac{1}{mc}\sum_{1\leqslant i<j\leqslant N}\delta''(x_i - x_j). \tag{2}$$

This pseudopotential can be used straightforwardly to first order in $1/c$. At second and higher orders, some divergences appear which can be cured by a point-splitting prescription [2]. In this paper, we will work only to first order in $1/c$ and will therefore not encounter any divergences.

We will be interested in the ground state (i.e. zero temperature) properties of the system in the thermodynamic limit in which $N, L \to \infty$, with the density

$$\rho_0 = \frac{N}{L} \tag{3}$$

held fixed. In this limit, we find, either from the exact solution [1] or from the pseudopotential approach [2], that the ground state energy per particle is given by

$$\frac{E_0}{N} = \frac{\pi^2\rho_0^2}{6m} - \frac{\pi^2\rho_0^3}{3mc} \tag{4}$$

to first order in $1/c$. At zero temperature, the chemical potential is given by

$$\mu = \left(\frac{\partial E_0}{\partial N}\right)_L = \frac{\pi^2\rho_0^2}{2m} - \frac{4\pi^2\rho_0^3}{3mc}. \tag{5}$$

For convenience, we define the Fermi momentum k_F and Fermi velocity v_F as

$$k_F = \pi\rho_0 \qquad \text{and} \qquad v_F = \frac{k_F}{m}. \tag{6}$$

We also define a dimensionless parameter

$$g = \frac{\rho_0}{c}. \tag{7}$$

Equations (4) and (5) imply that for large values of c, we have a system of fermions with a weak attractive interaction of strength $-g$.

The energy eigenstates of the system of non-interacting fermions ($g = 0$) have normalized wavefunctions which are given by $1/\sqrt{N!}$ times the Slater determinant of a matrix M. The entries of M are given by

$$M_{np} = \psi_n(x_p) \tag{8}$$

where $\psi_n(x)$ denotes the normalized one-particle wavefunctions. For the model on a circle, ψ_n is given by

$$\psi_n(x) = \frac{1}{\sqrt{L}}\,\mathrm{e}^{\mathrm{i}2\pi nx/L} \tag{9}$$

where $n = 0, \pm1, \pm2, \ldots$ (if N is odd), and the corresponding one-particle energies are $E_n = (2\pi n)^2/(2mL^2)$. For the system in a box with hard walls, the one-particle wavefunctions are given by

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi nx}{L}\right) \tag{10}$$

where $n = 1, 2, 3, \ldots$, and the corresponding energies are $E_n = (\pi n)^2/(2mL^2)$.

Using the pseudopotential given in (2), we can do perturbation theory to first order in $1/c$ as follows. For the non-interacting and weakly interacting systems, let us denote the N-particle wavefunctions of the energy eigenstates by $\Psi_n^{(0)}$ (which are normalized to unity) and Ψ_n respectively, and the corresponding energies by $E_n^{(0)}$ and E_n. To first order in $1/c$, we have

$$\Psi_n = \Psi_n^{(0)} + \sum_{l\neq n}\Psi_l^{(0)}\frac{\langle\Psi_l^{(0)}|V|\Psi_n^{(0)}\rangle}{E_n^{(0)} - E_l^{(0)}} \tag{11}$$

provided that there is no degeneracy at the energy $E_n^{(0)}$. (This will be true in our calculations since we will only apply (11) to the ground state which is unique both on the circle and in the box.) Note that the norm of Ψ_n in (11) differs from unity only by terms of order $1/c^2$ which we are going to ignore.

The one-particle density $\rho(x)$ is obtained from the many-particle wavefunction $\Psi(x_i)$ as

$$\rho(x) = N\int \mathrm{d}x_2\,\mathrm{d}x_3\ldots\mathrm{d}x_N\,\Psi^\star\Psi \tag{12}$$

where the factor of N has been introduced on the right-hand side to ensure that we obtain $\int_0^L \mathrm{d}x\, \rho(x) = N$. For the system on a circle, we find that the density in the ground state of N non-interacting fermions is simply equal to a constant, $\rho(x) = \rho_0$. For the system in a box, the density in the ground state of N non-interacting fermions is given by

$$\rho(x) = \sum_{n=1}^{N} \frac{2}{L} \sin^2\left(\frac{\pi n x}{L}\right). \tag{13}$$

In the thermodynamic limit, we then find that

$$\rho(x) = \rho_0 - \frac{\sin(2k_F x)}{2\pi x}. \tag{14}$$

The hole in the density integrates to $1/4$, namely,

$$\int_0^\infty \mathrm{d}x\, [\rho(x) - \rho_0] = -\frac{1}{4}. \tag{15}$$

The oscillations in the density in (14) are caused by reflection from the hard wall at $x = 0$. We will study below how these oscillations are modified by the interactions between the particles.

3. The δ-function Bose gas with a hard wall

We will now study the behaviour of the δ-function Bose gas with a hard wall. We will first consider the system placed in a box extending from $x = 0$ to $x = L$ with hard walls at both ends. We will then take the limit $L \to \infty$. Far away from the point $x = 0$, we will see that the leading order oscillatory term in the density has a logarithmic prefactor; the interpretation of this will be provided in section 6.

In the ground state $\Psi_0^{(0)}$ of the non-interacting system, the N particles occupy the lowest energy states described by (10) with $n = 1, 2, \ldots, N$. The perturbation in (2) will connect this state to two types of states:

(A) a state of the type $\Psi^{(0)}(n'; n)$ which differs from the ground state $\Psi_0^{(0)}$ in that only one particle is excited from the level n to the level n', where $1 \leqslant n \leqslant N$ and $n' > N$.

(B) a state of the type $\Psi^{(0)}(n'_1, n'_2; n_1, n_2)$ which differs from the ground state $\Psi_0^{(0)}$ in that only two particles are excited from the levels n_1 and n_2 to the levels n'_1 and n'_2, where $1 \leqslant n_2 < n_1 \leqslant N$ and $n'_1 > n'_2 > N$.

The matrix elements of (2) between $\Psi_0^{(0)}$ and states of type A are given by

$$V_{n';n} \equiv \left\langle \Psi^{(0)}(n'; n) | V | \Psi_0^{(0)} \right\rangle = \frac{\pi^2}{8mcL^3} A_{n';n}$$

where

$$\begin{aligned} A_{n',n} &= 3(n^2 + n'^2) + 10nn' && \text{if } n' + n \text{ is even and } 1 \leqslant \frac{n' + n}{2} \leqslant N \\ &= -3(n^2 + n'^2) + 10nn' && \text{if } n' + n \text{ is even and } 1 \leqslant \frac{n' - n}{2} \leqslant N \\ &= 0 && \text{otherwise.} \end{aligned} \tag{16}$$

The matrix elements of (2) between $\Psi_0^{(0)}$ and states of type B will be denoted by $V_{n'_1,n'_2;n_1,n_2}$; however, we do not need the expressions for these for reasons which will soon become clear.

Following (11), we see that the ground state wavefunction up to order $1/c$ is given by

$$\Psi_0 = \Psi_0^{(0)} + \sum_{n'>N} \sum_{1\leqslant n\leqslant N} \Psi^{(0)}(n';n) \frac{V_{n';n}}{(n^2-n'^2)\pi^2/(2mL^2)} + \sum_{n_1'>n_2'>N} \sum_{1\leqslant n_2<n_1\leqslant N} \Psi^{(0)}(n_1',n_2';n_1,n_2) \frac{V_{n_1',n_2';n_1,n_2}}{\left(n_1^2+n_2^2-n_1'^2-n_2'^2\right)\pi^2/(2mL^2)}. \tag{17}$$

We can now compute the density using (12). To order $1/c$, we see that we get a contribution from $\left\langle \Psi_0^{(0)} | \psi_0^{(0)} \right\rangle$ which is given in (14), and a contribution from the cross-terms $\left\langle \Psi_0^{(0)} \middle| \Psi^{(0)}(n';n) \right\rangle$ arising from the type A states in (17). There is *no* contribution from the cross-terms arising from the type B states since the integration over $\mathrm{d}x_2 \ldots \mathrm{d}x_N$ ensures that $\left\langle \Psi_0^{(0)} \middle| \Psi^{(0)}(n_1',n_2';n_1,n_2) \right\rangle$ does not contribute to the density. The vanishing of the contribution of type B states leads to a major simplification in the calculations.

We now find that

$$\rho(x) - \rho_0 + \frac{\sin(2k_F x)}{2\pi x} = \frac{1}{cL^2} \sum_{n',n} \sin\left(\frac{\pi n' x}{L}\right) \sin\left(\frac{\pi n x}{L}\right) \frac{A_{n';n}}{n^2-n'^2} \tag{18}$$

where $A_{n';n}$ is given in (16). We now go to the thermodynamic limit. Let us introduce the variables $k = \pi n/L$ and $k' = \pi n'/L$, so that $0 \leqslant k \leqslant k_F$ and $k' \geqslant k_F$. We replace the sums over n and n' by $(L/\pi)\int \mathrm{d}k$ and $(L/\pi)\int \mathrm{d}k'$, respectively. The condition for non-vanishing matrix elements in (16), namely, that $n'+n$ must be an even integer, implies that we should put a factor of $1/2$ in front of the integrals over k and k' . Further, the conditions that $(n' \pm n)/2$ must be an integer lying in the range 1 to N turn into the conditions $0 \leqslant k' \pm k \leqslant 2k_F$. After some manipulations, we find that (18) can be written in the form

$$\rho(x) - \rho_0 + \frac{\sin(2k_F x)}{2\pi x} = -\frac{1}{2\pi^2 c} \int_{-k_F}^{k_F} \mathrm{d}k \int_{\pi\rho_0}^{2k_F-k} \mathrm{d}k' \frac{\sin(kx)\sin(k'x)}{k'^2-k^2} [3(k'^2+k^2)+10k'k]. \tag{19}$$

We now define two new variables $u = (k'+k)/(2k_F)$ and $v = (k'-k)/(2k_F)$. In terms of these variables, the limits of the integrals over k and k' in (19) are equivalent either to $\int_0^1 \mathrm{d}u \int_{1-u}^{1+u} \mathrm{d}v$ or to $\int_0^2 \mathrm{d}v \int_{|v-1|}^1 \mathrm{d}u$. Finally, we can write $\sin(kx)\sin(k'x) = \frac{1}{2}[\cos(2k_F xv) - \cos(2k_F xu)]$. We can now do one of the integrals, either over u or over v, to obtain

$$\rho(x) - \rho_0 + \frac{\sin(2k_F x)}{2\pi x} = -\rho_0 g \left[\int_0^2 \mathrm{d}v \cos(2k_F xv) \left\{ 2 - v + \frac{v}{2} \ln|v-1| \right\} + \int_0^1 \mathrm{d}u \cos(2k_F xu) \left\{ 1 - 2u \ln\left(\frac{1+u}{1-u}\right) \right\} \right]. \tag{20}$$

As shown in the appendix, the total hole in the density is found to be

$$\int_0^\infty [\rho(x) - \rho_0] = -\frac{1}{4} - \frac{3g}{4}. \tag{21}$$

From (20), we can obtain the asymptotic form of the density at large values of x (i.e. for $|k_F x| \gg 1$) as shown in the appendix. The density turns out to have an expansion in powers of $1/x$ multiplied by sines, cosines and logarithms. The leading order terms are given by

$$\rho(x) = \rho_0 - \frac{\sin(2k_F x)}{2\pi x} [1 - 2g \ln(4\pi\, \mathrm{e}^{C-1/2} \rho_0 x)] + 3g \frac{\cos(2k_F x)}{4x} \tag{22}$$

where $C = 0.5772\ldots$ is Euler's constant. We will see in section 6 that the logarithm is a sign that the power of x in the denominator of $\sin(2k_F x)$ should really be $1 + 2g$ (plus terms of higher order in g), rather than 1. Namely,

$$\frac{1}{(\rho_0 x)^{1+2g}} = \frac{1}{\rho_0 x}[1 - 2g\ln(\rho_0 x)] \tag{23}$$

plus terms of higher order in g.

4. The δ-function Bose gas with a discontinuity in the interaction strength

In this section, we will consider another application of the pseudopotential approach. We will consider a δ-function Bose gas on a circle, with the interaction parameter in (1) being equal to a large value c for $0 < x < L/2$ and ∞ for $L/2 < x < L$. It is clear from (5) that the chemical potential will then be different in the two halves of the system; this would imply that a state which has the same density everywhere can no longer be close to the ground state. We will compensate for this imbalance by adding a one-body potential in the region $0 < x < L/2$ which is equal to

$$\delta V = \frac{4\pi^2\rho_0^3}{3mc}. \tag{24}$$

This ensures that the chemical potential and the density are the same in the two halves of the system; therefore, the ground states of the non-interacting ($c = \infty$) and interacting (large c) systems are smoothly connected to each other. We will therefore work with the following perturbation to the non-interacting system:

$$V = -\frac{1}{mc}\sum_{1\leqslant i<j\leqslant N}\delta''(x_i - x_j)f(x_i) + \frac{4\pi^2\rho_0^3}{3mc}\sum_{1\leqslant i\leqslant N} f(x_i)$$

where

$$\begin{aligned} f(x) &= 1 \qquad \text{for } 0 < x < L/2 \\ &= 0 \qquad \text{for } L/2 < x < L. \end{aligned} \tag{25}$$

We will then find that there is a reflection from the point of discontinuity at $x = 0$.

We will now compute the density of the system to first order in $1/c$. In the ground state $\Psi_0^{(0)}$ of the non-interacting system, the particles occupy the levels given in (9) with $n = 0, \pm1, \pm2, \ldots, \pm(N-1)/2$ (assuming that N is odd). As in section 3, the perturbation in (25) connects this state $\Psi_0^{(0)}$ to states of types A and B. Type A states differ from $\Psi_0^{(0)}$ in that only one particle is excited from a level n lying in the range $[-(N-1)/2, (N-1)/2]$ to a level n' lying outside that range, while type B states differ from $\Psi_0^{(0)}$ in that only two particles are excited from levels n_1 and n_2 lying in the range $[-(N-1)/2, (N-1)/2]$ to levels n_1' and n_2' lying outside that range. After using (11) to write the perturbed ground state wavefunction to order $1/c$, we again find that the type B states do not contribute to the density.

The matrix element of (25) between $\Psi_0^{(0)}$ and a type A state $\Psi^{(0)}(n'; n)$ is given by

$$\begin{aligned} V_{n';n} &\equiv \left\langle \Psi^{(0)}(n'; n)|V|\Psi_0^{(0)}\right\rangle \\ &= \frac{\mathrm{i}4k_F}{mcL^2(n'-n)}\left[n'n + \frac{N^2-1}{12} - \frac{\rho_0^2L^2}{3}\right] \qquad \text{if } n'-n \text{ is odd} \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{26}$$

We now go to the thermodynamic limit, and introduce the variables $k' = 2\pi n'/L$ and $k = 2\pi n/L$. Thus k lies in the range $[-k_F, k_F]$ while k' lies outside it. We replace the

sums over n' and n by the integrals $(L/2\pi)\int \mathrm{d}k'$ and $(L/2\pi)\int \mathrm{d}k$, respectively, and put a factor of $1/2$ in front of the integrals to take care of the restriction in (26) that $n'-n$ must be an odd integer. We then find that the density is given by

$$\rho(x)=\rho_0+\frac{g}{\pi^2}\int_{-k_F}^{k_F}\mathrm{d}k\int_{|k'|\geqslant k_F}\mathrm{d}k'\frac{\sin\{(k'-k)x\}}{(k'^2-k^2)(k'-k)}\left[k'k-k_F^2\right]. \tag{27}$$

From this expression, it is clear that the choice of the one-body potential in (24) makes the matrix element for forward scattering ($k'=k=k_F$ or $k'=k=-k_F$) vanish. This will lead to the result below that $\rho(x)-\rho_0$ vanishes for $x\to\pm\infty$; this is the reason for choosing the value of δV as given in (24).

We now introduce the variables $u=(k'+k)/(2k_F)$ and $v=(k'-k)/(2k_F)$. For $v\geqslant 0$, u lies in the range $|v-1|$ to $v+1$, while for $v\leqslant 0$, u lies in the range $v-1$ to $-|v+1|$. After integrating over u, we obtain

$$\rho(x)=\rho_0+\frac{g}{2\pi}\int_0^\infty \mathrm{d}v\frac{\sin(2k_Fxv)}{v^2}\left[2v-(v^2+1)\ln\left(\frac{v+1}{|v-1|}\right)\right]. \tag{28}$$

From (28), we can obtain the asymptotic behaviour of the density as $x\to\pm\infty$ using the methods given in the appendix. We find that

$$\begin{aligned}\rho(x)&=\rho_0-g\frac{\sin(2k_Fx)}{2\pi x}\qquad &&\text{for } x\to\infty\\ &=\rho_0+g\frac{\sin(2k_Fx)}{2\pi x}\qquad &&\text{for } x\to-\infty.\end{aligned} \tag{29}$$

We will now see that the oscillatory terms in (29) can be interpreted as being due to reflection of the particles from the point $x=0$. Consider a system in which fermions coming in from $x=\infty$ get reflected back from $x=0$ with an amplitude r_+. The density at large positive values of x can be shown to be of the form [7, 8]

$$\rho(x)=\rho_0+\frac{\mathrm{i}}{4\pi x}[r_+^\star\,\mathrm{e}^{-\mathrm{i}2k_Fx}-r_+\,\mathrm{e}^{\mathrm{i}2k_Fx}]. \tag{30}$$

On comparing this with (29), we see that

$$r_+=-g. \tag{31}$$

Similarly, if fermions coming in from $x=-\infty$ get reflected back at $x=0$ with an amplitude r_-, the density at large negative values of x can be shown to be

$$\rho(x)=\rho_0+\frac{\mathrm{i}}{4\pi x}[r_-\,\mathrm{e}^{-\mathrm{i}2k_Fx}-r_-^\star\,\mathrm{e}^{\mathrm{i}2k_Fx}]. \tag{32}$$

On comparing this with (29), we see that

$$r_-=g. \tag{33}$$

In section 6, we will see that the values of r_+ and r_- obtained here agree exactly with those obtained from bosonization.

5. The δ-function Bose gas with a non-zero pseudopotential in a finite region

We will now extend the analysis of the previous section to the case where the interaction parameter c is non-zero over a finite length l, where l will be held fixed as we go to the thermodynamic limit $L\to\infty$. Now there is no particular reason to choose the strength of the one-body potential δV in the finite region $0<x<l$ to be the same as in (24); as long as δV is small, the ground state of this system is smoothly connected to that of the non-interacting system in which $1/c$ and δV are zero everywhere. At the end of our analysis, we will discover

that the reflection from the finite region is zero, i.e. there is a resonance in the transmission, if either l has certain special values (given by $\pi n/k_F$), or if δV has a particular value.

We will work with the following perturbation to the non-interacting ($c = \infty$) system:

$$V = -\frac{1}{mc}\sum_{1\leqslant i<j\leqslant N}\delta''(x_i - x_j)f(x_i) + \delta V\sum_{1\leqslant i\leqslant N} f(x_i) \tag{34}$$

where $f(x) = 1$ for $0 < x < c, = 0$ for $c < x < L$. As in section 4, we will compute the density of the system to first order in $1/c$. Once again, we find that the perturbation in (34) connects the ground state $\Psi_0^{(0)}$ of the non-interacting system to states of types A and B, and the type B states do not contribute to the density to order $1/c$.

The matrix element of (34) between $\Psi_0^{(0)}$ and a type A state $\Psi^{(0)}(n'; n)$ is given by

$$\begin{aligned}\left\langle\Psi^{(0)}(n'; n)|V|\Psi_0^{(0)}\right\rangle \\ = \frac{\mathrm{i}4k_F}{L^2(n'-n)}\left[\frac{1}{mc}\left(n'n + \frac{N^2-1}{12}\right) - \frac{L^2\delta V}{4\pi^2\rho_0}\right]\left[\exp\left\{\mathrm{i}2\pi(n-n')\frac{l}{L}\right\} - 1\right].\end{aligned} \tag{35}$$

We introduce the variables $k = 2\pi n/L$ lying in the range $[-k_F, k_F]$ and $k' = 2\pi n'/L$ lying outside that range. After going to the thermodynamic limit, we find that the density is given by

$$\begin{aligned}\rho(x) = \rho_0 - \frac{g}{\pi^2}\int_{-k_F}^{k_F}\mathrm{d}k\int_{|k'|\geqslant k_F}\mathrm{d}k'\left[\frac{\sin\{(k'-k)(x-l)\} - \sin\{(k'-k)x\}}{(k'^2-k^2)(k'-k)}\right] \\ \times\left[k'k + \frac{\pi^2\rho_0^2}{3} - \frac{mc\delta V}{\rho_0}\right].\end{aligned} \tag{36}$$

After introducing the variables $u = (k'+k)/(2k_F)$ and $v = (k'-k)/(2k_F)$, and integrating over u, we obtain

$$\begin{aligned}\rho(x) = \rho_0 - \frac{g}{2\pi}\int_0^\infty \mathrm{d}v\left[\frac{\sin\{(2k_F(x-l)v\} - \sin\{2k_Fxv\}}{v^2}\right] \\ \times\left[2v - \left(v^2 - \frac{1}{3} + \frac{mc\delta V}{\pi^2\rho_0^3}\right)\ln\left(\frac{v+1}{|v-1|}\right)\right].\end{aligned} \tag{37}$$

Using the methods given in the appendix, we find that the asymptotic behaviour of the density as $x \to \pm\infty$ is given by

$$\begin{aligned}\rho(x) &= \rho_0 + \left(\frac{2\rho_0}{3c} + \frac{m\delta V}{\pi^2\rho_0^2}\right)\frac{\sin\{2k_F(x-l)\} - \sin\{2k_Fx\}}{4\pi x} \qquad \text{for } x\to\infty \\ &= \rho_0 - \left(\frac{2\rho_0}{3c} + \frac{m\delta V}{\pi^2\rho_0^2}\right)\frac{\sin\{2k_F(x-l)\} - \sin\{2k_Fx\}}{4\pi x} \qquad \text{for } x\to-\infty.\end{aligned} \tag{38}$$

As in section 4, the oscillatory terms in (38) can be interpreted as being due to reflection of the particles from the region $0 < x < l$. On comparing (38) with the expressions in (30) and (32), we find that

$$r_+ = -\mathrm{i}\left(\frac{2\rho_0}{3c} + \frac{m\delta V}{\pi^2\rho_0^2}\right)\sin(k_Fl)\,\mathrm{e}^{-\mathrm{i}k_Fl} \tag{39}$$

and

$$r_- = -\mathrm{i}\left(\frac{2\rho_0}{3c} + \frac{m\delta V}{\pi^2\rho_0^2}\right)\sin(k_Fl)\,\mathrm{e}^{\mathrm{i}k_Fl}. \tag{40}$$

We observe that the reflection amplitudes are zero (and the transmission probabilities are therefore equal to 1) if either

$$l = \frac{\pi n}{k_F} \tag{41}$$

where n is an integer, or

$$\delta V = -\frac{2\pi^2 \rho_0^3}{3mc}. \tag{42}$$

We see that up to order $1/c$, the resonance condition in (41) is the same as for non-interacting fermions ($c = \infty$); equation (41) is familiar from scattering theory in one-particle quantum mechanics. The condition in (42) appears to be new; note that if (42) is satisfied, then $r_\pm$ is zero for all values of the length l. From (36), we see that the resonance conditions in (41) and (42) imply that the matrix element for backward scattering ($k' = -k = k_F$ or $k' = -k = -k_F$) is zero.

6. The Tomonaga–Luttinger liquid approach to the δ-function Bose gas

We will now discuss how the results obtained in sections 3 and 4 may be understood using the TLL description of the long-wavelength properties of a δ-function Bose gas. As mentioned in section 1, a TLL is described by the parameters k_F, v and K. These parameters can be deduced if one knows the energy and momentum of the low-lying states [4]. Let us assume that the model is defined on a circle of length L with periodic boundary conditions. Let the energy of the ground state be $E_0(N, L)$. Then one has the relation

$$\left(\frac{\partial^2 E_0}{\partial N^2}\right)_L = \frac{\pi v}{LK}. \tag{43}$$

Next, for a model which is invariant under Galilean transformations (as is true for the δ-function Bose gas), it turns out that

$$vK = v_F. \tag{44}$$

Since the ground state energy of the δ-function Bose gas is exactly known [1], one can use (43) and (44) to find the values of v and K for any value of the interaction parameter c. From (4), we find that

$$\begin{aligned} K &= 1 + 2g \\ v &= v_F(1 - 2g) \end{aligned} \tag{45}$$

up to order g.

The low-energy and long-wavelength properties of a one-dimensional fermionic system are determined by the modes near the two Fermi points with momenta $\pm k_F$. The second-quantized fermion fields Ψ can be written in terms of the fields near the two Fermi points as

$$\Psi(x) = \Psi_R(x)\, \mathrm{e}^{\mathrm{i}k_F x} + \Psi_L(x)\, \mathrm{e}^{-\mathrm{i}k_F x} \tag{46}$$

where the subscripts R and L denote right-moving, and left-moving, respectively. Let us define the density operators for these two fields as $\hat{\rho}_R = \Psi_R^\dagger \Psi_R$ and $\hat{\rho}_L = \Psi_L^\dagger \Psi_L$. Consider a contact interaction of the form [3, 4]

$$V_{\mathrm{int}} = \int \mathrm{d}x \left[g_2 \hat{\rho}_R(x) \hat{\rho}_L(x) + \frac{g_4}{2} \left\{ \hat{\rho}_R^2(x) + \hat{\rho}_L^2(x) \right\} \right]. \tag{47}$$

Then one can show that

$$K = \left[\left(v_F + \frac{g_4}{2\pi} - \frac{g_2}{2\pi}\right)\Big/\left(v_F + \frac{g_4}{2\pi} + \frac{g_2}{2\pi}\right)\right]^{1/2}$$
$$v = \left[\left(v_F + \frac{g_4}{2\pi} - \frac{g_2}{2\pi}\right)\left(v_F + \frac{g_4}{2\pi} + \frac{g_2}{2\pi}\right)\right]^{1/2}. \tag{48}$$

Comparing these expressions with (45), we see that for the δ-function Bose gas near the fermionic limit,

$$\frac{g_2}{2\pi v_F} = \frac{g_4}{2\pi v_F} = -2g. \tag{49}$$

We can directly verify the expression for g_2 given in (49). Consider a density–density interaction of the form

$$V_{\text{int}} = \frac{1}{2}\int\int \mathrm{d}x\,\mathrm{d}y\hat{\rho}(x)U(x-y)\hat{\rho}(y) \tag{50}$$

where the density operator follows from (46),

$$\begin{aligned}\hat{\rho}(x) &= \Psi^\dagger(x)\Psi(x)\\ &= \Psi_R^\dagger(x)\Psi_R(x) + \Psi_L^\dagger(x)\Psi_L(x) + \Psi_R^\dagger(x)\Psi_L(x)\,\mathrm{e}^{-\mathrm{i}2k_F x} + \Psi_L^\dagger(x)\Psi_R(x)\,\mathrm{e}^{\mathrm{i}2k_F x}.\end{aligned} \tag{51}$$

If the two-body potential $U(x)$ has a short range, a comparison of (47) and (50) shows that the parameter g_2 is related to the Fourier transform of $U(x)$ as

$$g_2 = \tilde{U}(0) - \tilde{U}(2k_F). \tag{52}$$

Comparing (50) and (2), we see that

$$U(x) = -\frac{1}{mc}\delta''(x). \tag{53}$$

Then (52) implies that

$$g_2 = -\frac{4k_F^2}{mc} \tag{54}$$

which agrees with (49).

Now we consider the bosonized form of the TLL theory [3, 4]. The bosonic field $\phi(x,t)$ is governed by the Lagrangian density

$$\mathcal{L} = \frac{1}{2vK}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{v}{2K}\left(\frac{\partial\phi}{\partial x}\right)^2. \tag{55}$$

The equations of motion are given by $\partial^2\phi/\partial t^2 = v^2\partial^2\phi/\partial x^2$. The excitations of the system therefore have the dispersion $\omega = v|k|$.

The technique of bosonization relates the Fermi fields Ψ_R and Ψ_L to exponentials of the second-quantized boson field ϕ. The exact relationship between the two fields depends on the geometrical situation. Let us consider the model of section 3, where the system is defined on the half-line $x \geqslant 0$. We denote the incoming (left-moving) Fermi field as Ψ_L and the outgoing (right-moving) Fermi field as Ψ_R. These are not independent fields since one is related to the other by reflection at $x = 0$. One can now 'unfold' the half-line to the full line and define all the fields to be purely left-moving [3]. The Fermi fields $\tilde{\Psi}_L$ on the full line are related to those on the half-line as

$$\tilde{\Psi}_L(x) = \Psi_L(x) \qquad \tilde{\Psi}_L(-x) = \Psi_R(x) \tag{56}$$

where $x > 0$. The bosonized form of this fermionic theory also contains only a left-moving boson field ϕ_L defined on the full line; the two fields are related as

$$\tilde{\Psi}_L(x) \sim \exp\left[\mathrm{i}\sqrt{\frac{\pi}{K}}\{\phi_L(x) + \phi_L(-x)\} + \mathrm{i}\sqrt{\pi K}\{\phi_L(x) - \phi_L(-x)\}\right]. \tag{57}$$

Now we can compute the fermion density which is equal to an expectation value in the ground state, $\rho(x) = \langle\Psi^\dagger(x)\Psi(x)\rangle$, where $x > 0$. Following (51) and (56), this is given by

$$\begin{aligned}\rho(x) = {} & \left\langle\tilde{\Psi}_L^\dagger(x)\tilde{\Psi}_L(x)\right\rangle + \left\langle\tilde{\Psi}_L^\dagger(-x)\tilde{\Psi}_L(-x)\right\rangle \\ & + \left\langle\tilde{\Psi}_L^\dagger(-x)\tilde{\Psi}_L(x)\right\rangle \mathrm{e}^{-\mathrm{i}2k_F x} + \left\langle\tilde{\Psi}_L^\dagger(x)\tilde{\Psi}_L(-x)\right\rangle \mathrm{e}^{\mathrm{i}2k_F x}.\end{aligned} \tag{58}$$

On using the bosonization expression (57), we find that the first two terms on the right-hand side of (58) are independent of x; they give rise to a constant which is ρ_0. The last two terms in (58) give

$$\langle\exp[\pm\,\mathrm{i}2\sqrt{\pi K}\{\phi_L(-x) - \phi_L(x)\}]\rangle \sim \frac{1}{x^K}. \tag{59}$$

Including the factors of $\mathrm{e}^{\pm\,\mathrm{i}2k_F x}$ in (58), we find that

$$\rho(x) - \rho_0 \sim \frac{\sin(2k_F x)}{x^K}. \tag{60}$$

We thus see that in the presence of a hard wall, the density of a TLL far from the wall has an oscillatory piece whose amplitude decays as $1/x^K$ [6]. If K is close to 1 as in (45), we see that the amplitude has an expansion in powers of g which is given by $(1 - 2g\ln x)/x$. This is exactly what we found in section 3.

We now turn to the model of section 4, where the Luttinger parameters are given by (v, K) for $x > 0$ and by $(v_F, 1)$ for $x < 0$. We can use the Lagrangian density in (55) with these parameters to find the equations of motion for the boson fields. The matching conditions at $x = 0$ turn out to be [5]

$$\phi(x = 0-, t) = \phi(x = 0+, t) \qquad v_F\left(\frac{\partial\phi(x,t)}{\partial x}\right)_{x=0-} = \frac{v}{K}\left(\frac{\partial\phi(x,t)}{\partial x}\right)_{x=0+}. \tag{61}$$

We can now consider what happens when a wave is incident from $x = -\infty$. The equations of motion give

$$\begin{aligned}\phi(x,t) &= \exp[\mathrm{i}k(x - v_F t)] + r_-\exp[-\mathrm{i}k(x + v_F t)] \qquad \text{for } x < 0 \\ &= t_-\exp\left[\mathrm{i}\frac{kv_F}{v}(x - vt)\right] \qquad \text{for } x > 0.\end{aligned} \tag{62}$$

The matching conditions in (61) now lead to the following expressions for the reflection and transmission amplitudes:

$$r_- = \frac{K-1}{K+1} \qquad t_- = \frac{2K}{K+1}. \tag{63}$$

Note that current conservation is satisfied since $v_F(1 - r_-^2) = vt^2$ and $v_F = vK$. Similarly, for a wave incident from $x = \infty$, we find that

$$r_+ = \frac{1-K}{K+1} \qquad t_+ = \frac{2}{K+1} \tag{64}$$

which satisfies current conservation since $v\left(1 - r_+^2\right) = v_F t_+^2$. Upon using (45), we see that the reflection amplitudes $r_\pm$ obtained here agree with those obtained in section 4.

We therefore have the interesting result that a discontinuity in the interaction parameters is sufficient to cause scattering, even if there is no other scattering mechanism (such as an

impurity) present in the system. The calculations presented in section 4 can be viewed as a microscopic derivation of this interesting phenomenon which had earlier been obtained only from bosonization [5].

The results in section 5 have implications for the subject of transport through a quantum wire which is sometimes modelled as a TLL of finite length which is bounded on both sides by Fermi liquid leads [5, 9, 10]. In these models, therefore, the Luttinger parameters K and v change discontinuously at the contacts between the quantum wire and its leads. For the case of two identical impurities in a TLL, it is known that the transmission resonances are infinitely sharp at zero temperature [10]. Although the model we have studied in section 5 has two points of discontinuity in the interaction parameter, rather than two impurities, it is possible that the structure of the transmission resonance will be found to change significantly if we go up to higher orders in the interaction parameter $1/c$.

7. Discussion

We have used the pseudopotential approach to study the behaviour of a system of fermions with weak attractive interactions. The various situations we have considered are not solvable by the Bethe ansatz [1]. This is because the Bethe ansatz only works in models which are both invariant under translations and have N commuting operators including the Hamiltonian. In such systems, the momenta of the N particles $k_1, k_2, \ldots, k_N$ are good quantum numbers, and the wavefunction can be found exactly as a superposition of $N!$ plane waves. In the absence of translation invariance, there are reflections (at a hard wall or at a point of discontinuity in the interaction strength) due to which the particle momenta are no longer good quantum numbers. The wavefunctions are then no longer a superposition of a finite number of plane waves, and they cannot be found exactly.

When the Bethe ansatz fails, the pseudopotential approach seems to be the only way to obtain exact results near the fermionic limit, although calculational difficulties may restrict its use to low orders in perturbation theory. We have shown how exact expressions for the density can be obtained in certain situations. While the agreement between our results and those obtained from bosonization for the asymptotic behaviour of the density is satisfying, we should also emphasize that there are relatively few models with interactions in which something can be computed at all distances.

Our methods can be applied to other problems involving weakly interacting fermions in one dimension. For instance, one can study the Kane–Fisher model of a single impurity placed in a TLL [10], and the effect of a junction of three or more semi-infinite wires [8]. While these problems have been studied earlier using bosonization (and other methods which are only valid at long wavelengths [7]), it may be interesting to apply the pseudopotential method to these situations since we may be able to obtain expressions for certain quantities which are valid at all distances. Finally, one can use the pseudopotential method to study dynamical quantities such as the conductance of a finite length TLL at finite frequencies.

After this work was completed, we found a paper which discusses some properties of the one-dimensional Bose–Hubbard model at low densities [11]; the continuum Hamiltonian which governs that system is essentially the same as the one studied by us.

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Appendix

In this appendix, we will discuss some methods for computing the various expressions for the density presented above.

Let us start with the expression for the hole in (21). Suppose that we have a function $f(x)$ defined by the integral

$$f(x) = \int_0^{u_0} \mathrm{d}u \cos(axu) h(u) \tag{A.1}$$

where $h(u)$ is finite and continuous at $u = 0$. Then we find

$$\begin{aligned}\int_0^\infty \mathrm{d}x f(x) &= \lim_{\alpha\to 0} \int_0^{u_0} \mathrm{d}u\, h(u) \int_0^\infty \mathrm{d}x \cos(axu)\, \mathrm{e}^{-\alpha x^2} \\ &= \lim_{\alpha\to 0} \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \int_0^{u_0} \mathrm{d}u\, h(u)\, \mathrm{e}^{-a^2u^2/(4\alpha)} \\ &= \frac{\pi}{2a} h(0). \end{aligned} \tag{A.2}$$

On applying this to (20), we obtain (21).

Next, we discuss how to obtain asymptotic expressions (for $ax \gg 1$) for functions of the type

$$\begin{aligned} f(x) &= \int_0^{u_0} \mathrm{d}u \cos(axu)\, h(u) \\ g(x) &= \int_0^{u_0} \mathrm{d}u \sin(axu)\, h(u). \end{aligned} \tag{A.3}$$

If $h(u)$ is finite and continuous for all values of u in the range $[0, u_0]$, then the leading order expressions for (A.3) are of order $1/x$, and they are obtained by integrating the functions $\cos(axu)$ and $\sin(axu)$. Namely,

$$\begin{aligned} f(x) &= \frac{1}{ax}[h(u_0)\sin(axu_0)] \\ g(x) &= \frac{1}{ax}[h(0) - h(u_0)\cos(axu_0)] \end{aligned} \tag{A.4}$$

plus terms of order $1/x^2$ and higher. These formulae are valid even if $u_0 = \infty$, provided that $h(\infty) = 0$.

Now suppose that the function $h(u)$ has a logarithmic divergence at one point. Some examples of this are the integrals [12]

$$\begin{aligned} \int_0^1 \mathrm{d}u \sin(axu) \ln u &= -\frac{1}{ax}[\ln(ax) + C] \\ \int_0^1 \mathrm{d}u \cos(axu) \ln u &= -\frac{\pi}{2ax} \end{aligned} \tag{A.5}$$

plus terms of order $1/x^2$ and higher. (Here $C = 0.5722\ldots$ is Euler's constant). From (A.5), one can show that

$$\begin{aligned} \int_0^1 \mathrm{d}u \sin(axu) \ln(1-u) &= \frac{1}{ax}\left[\ln(ax)\cos(ax) + C\cos(ax) - \frac{\pi}{2}\sin(ax)\right] \\ \int_0^1 \mathrm{d}u \cos(axu) \ln(1-u) &= -\frac{1}{ax}\left[\ln(ax)\sin(ax) + C\sin(ax) + \frac{\pi}{2}\cos(ax)\right] \end{aligned} \tag{A.6}$$

plus terms of order $1/x^2$ and higher.

Now consider a function $h(u)$ in (A.3) which contains a logarithmic divergence at $u = 0$ of the form $b \ln u$, where b is some constant. Then we define a new function $\tilde{h}(u) = h(u) - b \ln u$ which is finite for all values of u in the range of integration. We can then integrate over $\tilde{h}(u)$ using (A.3) and (A.4), and over $b \ln u$ using (A.5). Combining the two gives the result of integrating over $h(u)$. Similarly, we can compute the integrals in (A.3) if $h(u)$ has a logarithmic divergence at $u = u_0$. Finally, we can also compute integrals over functions $h(u)$ in (A.3) which have a logarithmic divergence at a point u_1 which lies inside the range $[0, u_0]$. We simply divide the integrals in (A.3) into two parts, one over the range $[0, u_1]$ and the other over the range $[u_1, u_0]$. Then each of these integrals can be computed as explained above. Using all these methods, one can derive the expressions in (22) and (29).

References

[1] Lieb E H and Liniger W 1963 *Phys. Rev.* **130** 1605
Lieb E H 1963 *Phys. Rev.* **130** 1616
[2] Sen D 1999 *Int. J. Mod. Phys.* A **14** 1789
[3] Gogolin A O, Nersesyan A A and Tsvelik A M 1998 *Bosonization and Strongly Correlated Systems* (Cambridge: Cambridge University Press)
[4] Haldane F D M 1981 *J. Phys. C: Solid State* **14** 2585
Schulz H J, Cuniberti G and Pieri P 2000 *Field Theories for Low-Dimensional Condensed Matter Systems* ed G Morandi, A Tagliacozzo and P Sodano (Berlin: Springer) (*Preprint* cond-mat/9807366)
Rao S and Sen D 2001 *Field Theories in Condensed Matter Systems* ed S Rao (New Delhi: Hindustan Book Agency) (*Preprint* cond-mat/0005492)
[5] Safi I and Schulz H J 1995 *Phys. Rev.* B **52** 17040
Safi I and Schulz H J 1999 *Phys. Rev.* B **59** 3040
[6] Cazalilla M A 2002 *Europhys. Lett.* **59** 793
[7] Yue D, Glazman L I and Matveev K A 1994 *Phys. Rev.* B **49** 1966
[8] Lal S, Rao S and Sen D 2002 *Phys. Rev.* B **66** 165327
[9] Maslov D L and Stone M 1995 *Phys. Rev.* B **52** 5539
Ponomarenko V V 1995 *Phys. Rev.* B **52** 8666
Furusaki A and Nagaosa N 1996 *Phys. Rev.* B **54** 5239
Lal S, Rao S and Sen D 2002 *Phys. Rev.* B **65** 195304 (*Preprint* cond-mat/0104402)
[10] Kane C L and Fisher M P A 1992 *Phys. Rev.* B **46** 15233
[11] Cazalilla M A 2003 *Preprint* cond-mat/0301167
[12] Gradshteyn I S and Ryzhik I M 1994 *Tables of Integrals, Series, and Products* (New York: Academic) p 612